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DESCRIPTION OF DETECTION PROBABILITY MODEL AND PROGRAM. (U)

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This is a working paper giving tentative information about some work in progress at NEL. If cited in the literature the information is to be identified as tentative and unpublished.

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L.K. Arndt

DESCRIPTION OF DETECTION PROBABILITY MODEL AND PROGRAM

INTRODUCTION

A program has been developed in Code 3110D to calculate the probability that at least N pulses in M consecutive pings will exceed a threshold as a function of the single ping signal to noise ratio, the cumulative false alarm rate and M. Before describing the multiple ping program a brief description of the program for the calculation of single ping detection probability will be given since it is a subroutine of the multi-ping program. Listings of both ETRAN* programs are included in the appendix.

SINGLE PING MODEL:

The single ping model is based upon the application of the Neyman-Pearson criterion to signal of unknown phase added to white Gaussian noise (ref. 2,3,4, 5,7,9). The Neyman-Pearson criterion places an upper limit on the probability of a false alarm and seeks to maximize the probability of detection within that limit. This criterion is in wide use in both radar and sonar work due to the difficulty in estimating prior probabilities and costs which are required by most other criteria.

The optimum receiver (i.e., matched filter) resulting from application of the Neyman-Pearson criterion to a signal with uniform random phase and additive white Gaussian noise is an envelope detector followed by a filter matched to the known envelope, as is well known (ref 2,3,4,5). Rice and others have derived the expressions for the probability of exceeding a set threshold with noise alone and with signal plus noise for this case (ref. 3,5). The density function for the signal plus noise is given by Rice as

$$p(v) = v \exp\left[-\frac{v^2+a^2}{2}\right] I_o(av)$$

where:

v = the random variable of output voltage normalized by the rms noise density from the matched filter

a = the voltage signal to noise ratio at the input to the detector. (signal amplitude/rms noise voltage)

$I_o(z)$ = the modified Bessel function of zero order and argument z.

*ETRAN is a scientific programming language developed by NEL Code 3110D used in Sonar Performance Prediction. Reference (11) describes this language.

The signal processing gain may be included in the expression by letting

$$a = (2E/N_o)^{1/2}$$

with E the signal energy and N_o the noise power density. This is an upper limit on the processing gain.

The Neyman-Pearson criterion says to pick an acceptable upper limit upon the probability of false alarm and maximize the probability of detection under that constraint. Let P_{fa} be the probability of false alarm and T* be the normalized voltage threshold, $T^* = [R^2/N_o]^{1/2}$, at the output of the detector. Then the probability that noise alone will exceed the threshold, R, is given by Rice's expression (ref 5) integrated from T* to infinity with a = 0:

$$P_{fa} = \int_{T^*}^{\infty} v \exp\left[-\frac{v^2}{2}\right] I_o(0)\, d\, v.$$

Making a change of variables $y = \frac{v^2}{2}$ and noting the $I_o(0) = 1$ gives

$$P_{fa} = \exp - (T^*)^2/2$$

Thus, specifying P_{fa} determines the threshold, T*, independent of the signal level (T* is a function of the noise power denisty).

The probability of detection (i.e., the probability that signal plus noise will exceed the threshold) is given by:

$$P_D = \int_{T^*}^{\infty} v \exp\left[-\frac{v^2+a^2}{2}\right] I_o(av)\, d\, v.$$

In general the integration cannot be performed analytically and curves giving P_D vs a for various values of P_{fd} have been generated by J.I. Marcum of the Rand Corporation (ref 9).

The single ping probability program calculates the probability of detection as a function of the probability of false alarm and the signal to noise ratio. For the cases of either the signal to noise ratio is much larger or much smaller than the normalized threshold, T^*, the asymptotic expressions from Helstrom (ref 3) are used:

for $a > T^* + 2.5$

$$P_D \approx 1 - \frac{(T^*/a)^{1/2}}{\sqrt{2\pi}\,(T^*-a)} \exp\left[-\frac{(a-T^*)^2}{2}\right].$$

for $a < T^* - 2.5$

$$P_D \approx \frac{(T^*/a)^{1/2}}{\sqrt{2\pi}\,(T^*-a)} \exp\left[-\frac{(a-T^*)^2}{2}\right]$$

For the case where the signal to noise ratio is close to the threshold T^* the modified Bessel function is approximated by the first three terms of its power series

$$I_o(z) \approx \frac{e^z}{\sqrt{2\pi z}} \left(1 + \frac{1}{8z} + \frac{3^2}{2!(8z)^2} + \frac{3^2 \cdot 5^2}{3!(8z)^3} + \cdots\right)$$

and using a sum to approximate the integral,

$$P_D \approx \sum_{v=T^*}^{N} \frac{1}{\sqrt{2\pi}} \left[1 + \frac{1}{8av} + \frac{9}{128(av)^2}\right] \left(\frac{v}{a}\right)^{1/2} \exp\left[-\frac{(v-a)^2}{2}\right] \Delta v$$

when

$$T^* - 2.5 \leq a \leq T^* + 2.5.$$

For false alarm probabilities between 10^{-3} and 10^{-12} and signal to noise ratios greater than two the differences between the results of Marcum and the model are less the 1%.

MULTIPLE PING DETECTION PROBABILITY

With the background of the single ping probability of detection calculations as a function of false alarm probability and signal to noise ratio, we shall now examine a different detection criterion based on M pings. The following assumptions are made for this first case:

a. The pings are independent of one another (i.e., the probability of two pings in a row exceeding the threshold is the product of the probability that each of the pings will exceed the threshold.)

b. The average signal to noise ratio is constant over the M pings.

c. The detection criterion is that N or more of the possible M pings exceed the threshold without regard to the order in which N occurs.

If the probability of exceeding the threshold on any one ping is p_s and of failing is $(1-p_s)$ then the probability of exactly N pulses out of M exceeding the threshold in a particular order is

$$P_1 = p_s^N (1-p_s)^{M-N}$$

Since the order of occurence is not important the probability of exactly N out of M in any order is just

$$P_2 = \binom{M}{N} P_1 = \binom{M}{N} p_s^N (1-p_s)^{M-N}$$

where $\binom{M}{N}$ is the binomial coefficient

$$\binom{M}{N} \equiv \frac{M!}{N!(M-N)!}$$

The probability of N <u>or more</u> threshold crossings out of M is the sum of the P_2's since the events are mutually exclusive.

$$\text{Prob } \{N \text{ or more } /M\} = \sum_{i=N}^{M} \binom{M}{i} p_s^i (1-p_s)^{M-i}$$

Using this equation, the probability of detection using a N or more out of M criterion may be calculated in terms of the single ping detection probability. Similarly the probability that noise alone will satisfy the criterion (i.e., a false alarm) may be calculated in terms of the single ping false alarm probability.

In applying the Neyman-Pearson criterion, we wish to specify some "N out of M" false alarm probability along with a single ping signal to noise ratio and calculate the "N out of M" detection probability. Since the equation for the output false alarm probability, F_o, in terms of the single ping false alarm rate, F_s, cannot be easily inverted an iterative solution is obtained by estimating F_s using the relation between the binomial distribution and the incomplete Beta function, $I_x(\alpha,\beta)$. An approximation to the inverse of the incomplete Beta function is given in ref. 1, page 945.

$$F_o = \sum_{i=N}^{M} \binom{M}{i} F_s^{\,i} (1-F_s)^{M-i}$$

$$F_o = I_{F_s} (N, M-N+1)$$

$$F_s \simeq \frac{N}{N+(M-N+1)\ e^{2w}} = \bar{F}_s$$

$$w \equiv \frac{y_p(h+\lambda)^{1/2}}{h} - \left(\frac{1}{2(M-N+1)-1} - \frac{1}{2N-1}\right)\left(\lambda + \frac{5}{6} - \frac{2}{3h}\right)$$

$$F_o \equiv \int_{y_p}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt$$

$$h \equiv 2\left(\frac{1}{2N-1} + \frac{1}{2(M-N+1)-1}\right)^{-1}$$

$$\lambda = \frac{y_p^{\,2}-3}{6}$$

The resulting estimate of F_s is used in the binomial sum to obtain $\overline{F}_o$. The error, $F_o - \overline{F}_o$, is further reduced using a correction equation of the form,

$$F_s = \overline{F}_s \left(\frac{F_o}{\overline{F}_o} \right)^{1/N}$$

This correction cycle may be iterated to obtain the required accuracy. Two corrections will normally give something less than 1% error in the cumulative false alarm probability. The resulting single ping false alarm probability, F_s, is used along with the single ping signal-to-noise ratio to obtain the single ping probability of detection as discussed in the first section. The P_s may then be used in the cumulative binomial to obtain the probability that the "N out of M" criterion is satisfied when the signal is present.

$$P_{N/M} = \sum_{i=N}^{M} \binom{M}{i} p_s^{\,i} (1-p_s)^{M-i}$$

The result of such a N or more out of M criterion is given in figures 1 through 5. Each figure shows the probability of detection vs the ratio of N/M with each curve representing a particular value of M between 2 and 12. Each figure is for a constant value of the single ping voltage signal-to-noise ratio shown as the value of A. The output probability of false alarm in all cases is approximately 10^{-7}.

The point to be made in examining this data is that for any fixed value of N and signal to noise, increasing M results in an increase in the probability of detection as would be expected since the available signal energy has been increased, but more importantly there is a "best" value of N for each M. An expression for estimating the best N as a function of M was given by Berkowitz in ref. 2.

$$N \approx .15 \sqrt{M}$$

For values of M between 2 and 12 the above expression results in a number within plus or minus one of the observed best value from the curves. The reduction

in the probability of detection from the peak value due to using a N from the above expression is small due to the slope of the curves changing slowly near the peak values.

Figure 6 presents some of the same data in a different form. The probability of detection is plotted vs M and as a function of the signal to noise ratio. In this case N was chosen as the "best" value for each M (i.e., N=3 for M=5). In going across the figure along same constant probability of detection the trade off between single ping signal to noise ratio and M is observed. (i.e., for $P_{M/N}=0.5$, M=1, N=1 requires $A \approx 3$.) In going up the figure along a particular value of M and N, the increase in probability of detection as a function of signal to noise is obtained. Finally along the lines of constant signal to noise ratio the increase in probability of detection vs M is obtained.

This analysis was based on the assumptions of a constant signal of unknown phase with additive white Gaussian noise. A less restrictive case of a signal with uniformly distributed phase and Gaussian distributed amplitude in additive white Gaussian noise may be solved using the same equations as above if the signal and noise are assumed independent and the signal fluctuations are assumed independent from pulse to pulse. Due to the independence between signal and noise and the assumed Gaussian distribution the probability distribution of the sum is also Gaussian with mean equal to the signal mean (noise mean assumed to be zero) and variances equal to the sum of the signal variances and the noise variances. This is the same form as the input for the case of constant signal except the variance is increased during the time the signal is present. Thus all previous results are carried over if an equivalent noise is defined as the square root of the sum of the variances and used any time the signal is present (not when determining the probability of false alarm). The assumption of a constant mean signal to noise ratio during the M pings may be relaxed in practice if the variation is small and the average value of the M single ping signal to noise ratios is used in the derived equations. It is hoped that even less restrictive cases (i.e., signal correlation between pings) will be programmed in the near future.

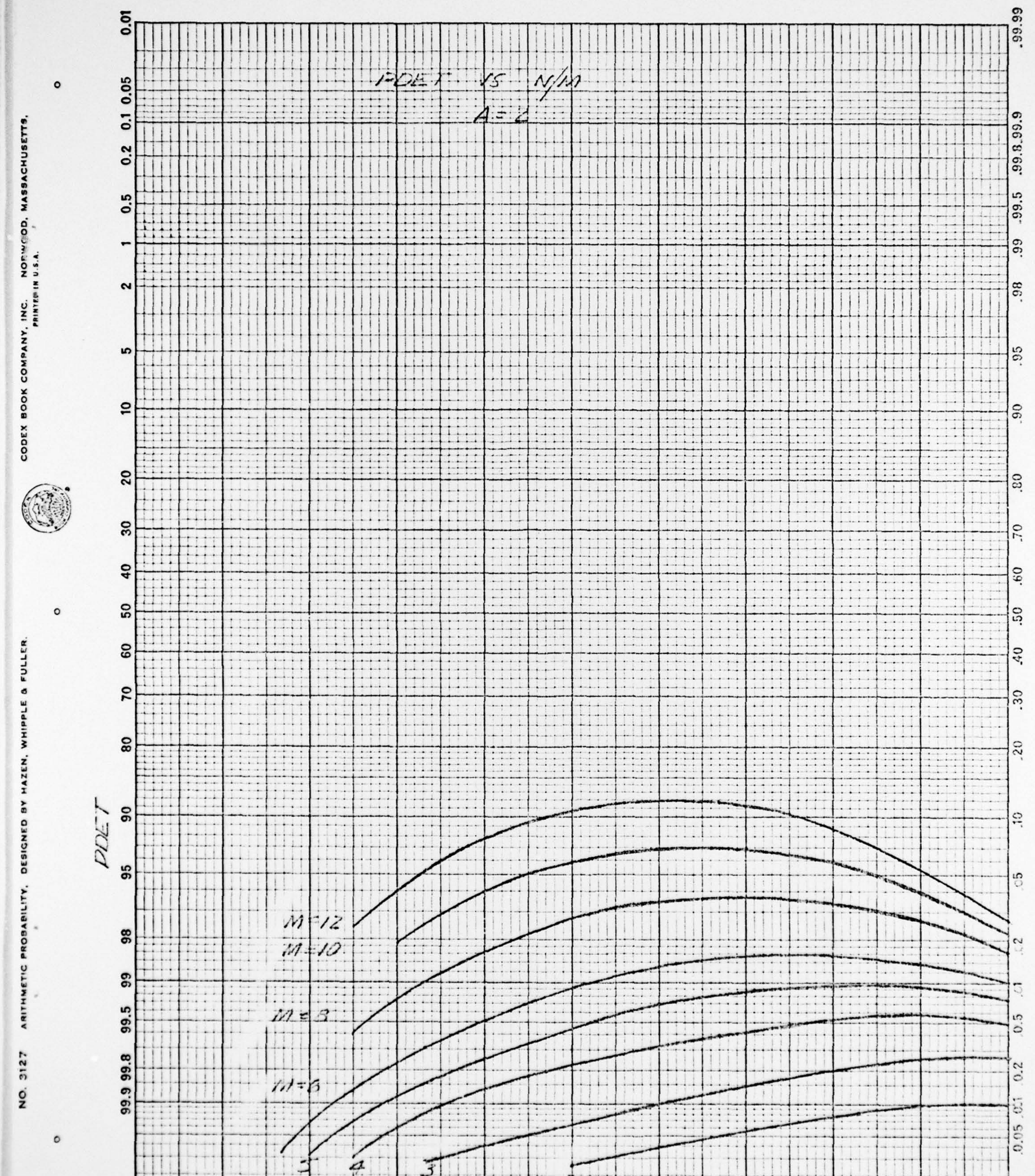

PDET VS N/M
A=2
PDET
M=12
M=10
M=8
M=6
5
4
3
2
N/M
NO. 3127 ARITHMETIC PROBABILITY, DESIGNED BY HAZEN, WHIPPLE & FULLER.
CODEX BOOK COMPANY, INC. NORWOOD, MASSACHUSETTS,
PRINTED IN U.S.A.

Figure 1

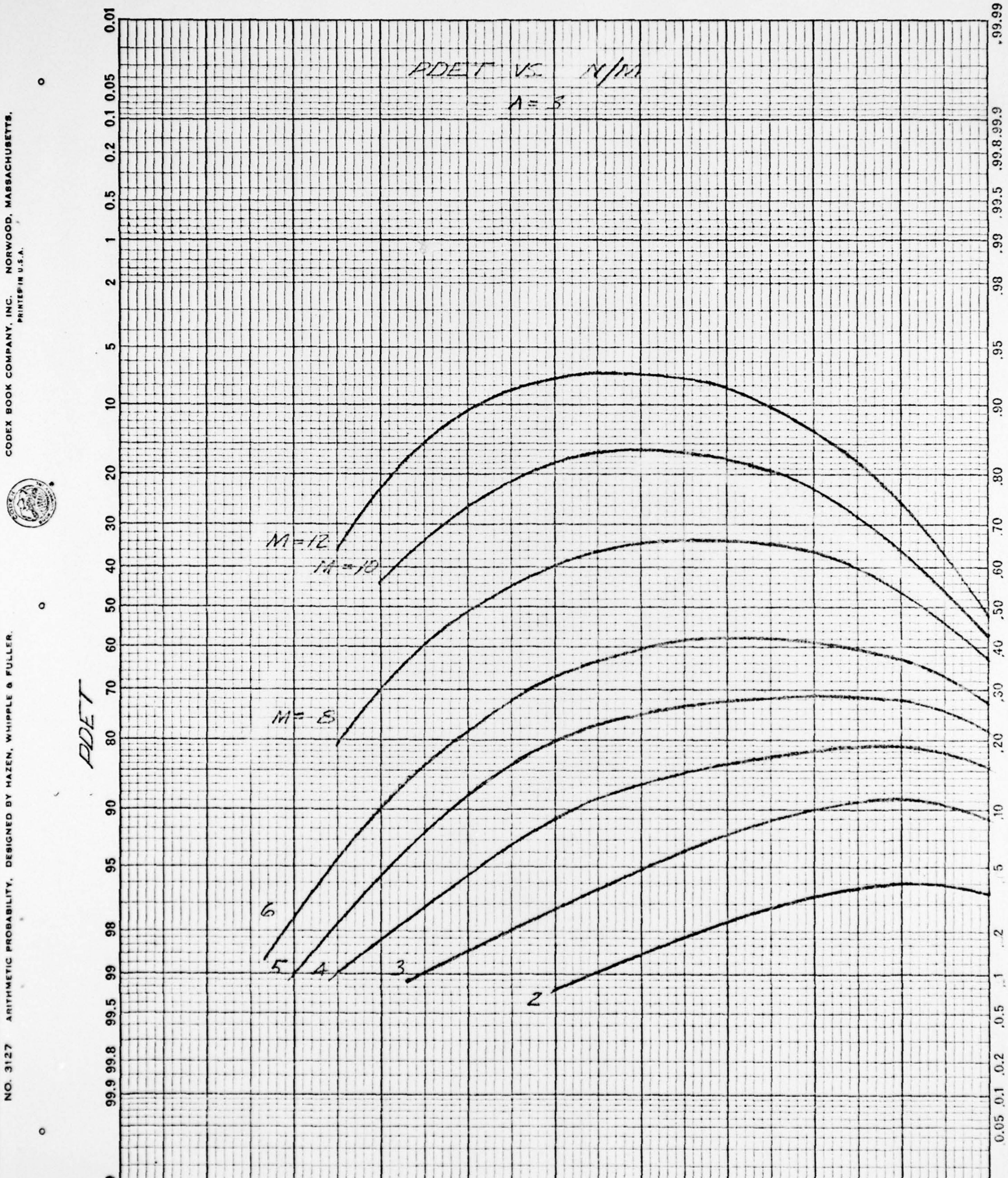

PDET VS. N/M
A = 3
M=12
M=10
M=8
6
5
4
3
2
PDET
N/M
CODEX BOOK COMPANY, INC. NORWOOD, MASSACHUSETTS. PRINTED IN U.S.A.
NO. 3127 ARITHMETIC PROBABILITY. DESIGNED BY HAZEN, WHIPPLE & FULLER.

Figure 2

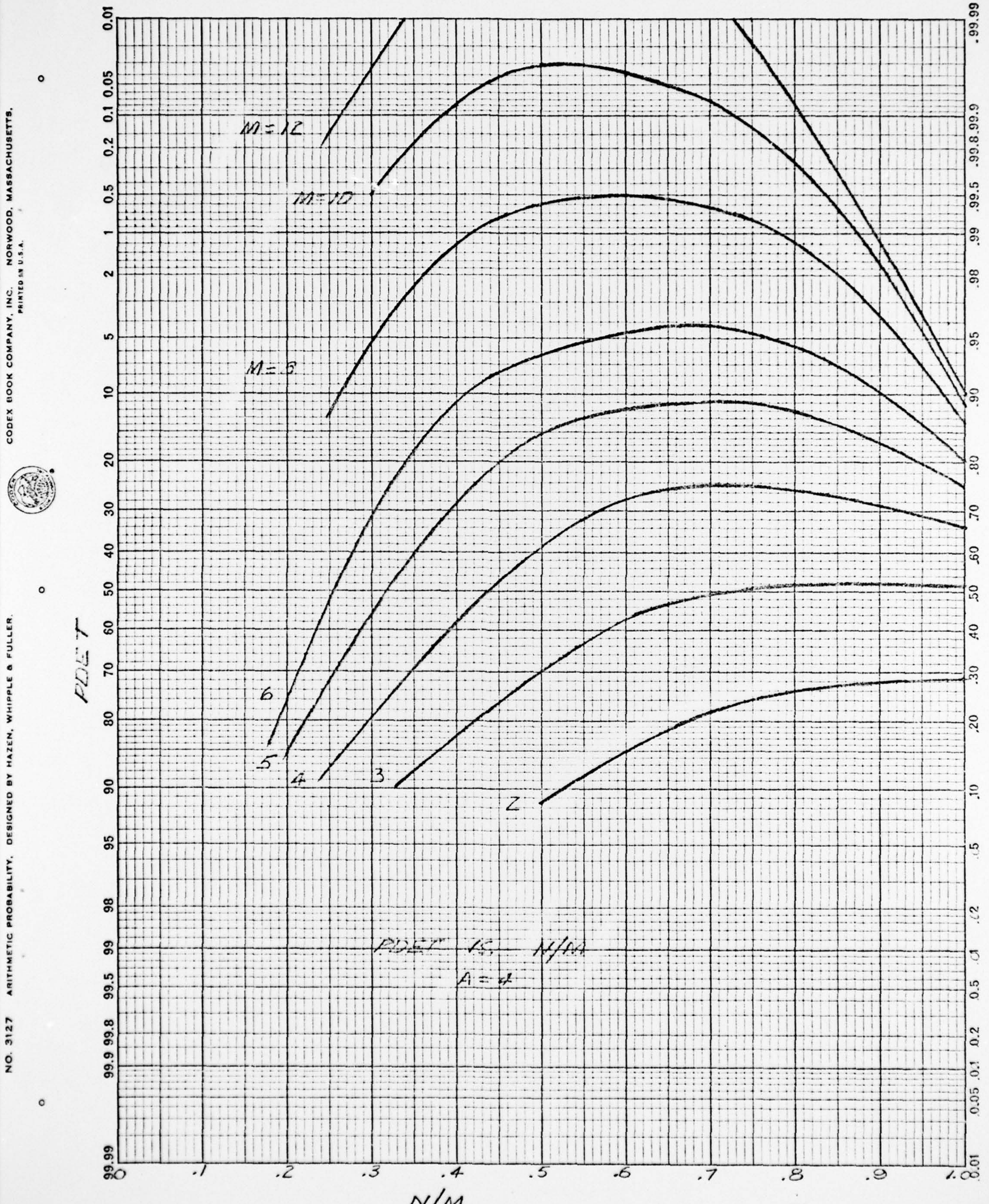

NO. 3127 ARITHMETIC PROBABILITY. DESIGNED BY HAZEN, WHIPPLE & FULLER.
CODEX BOOK COMPANY, INC. NORWOOD, MASSACHUSETTS.
PRINTED IN U.S.A.
M=12
M=10
M=8
6
5
4
3
2
PDET
PDET VS. N/M
A=4
N/M

Figure 3

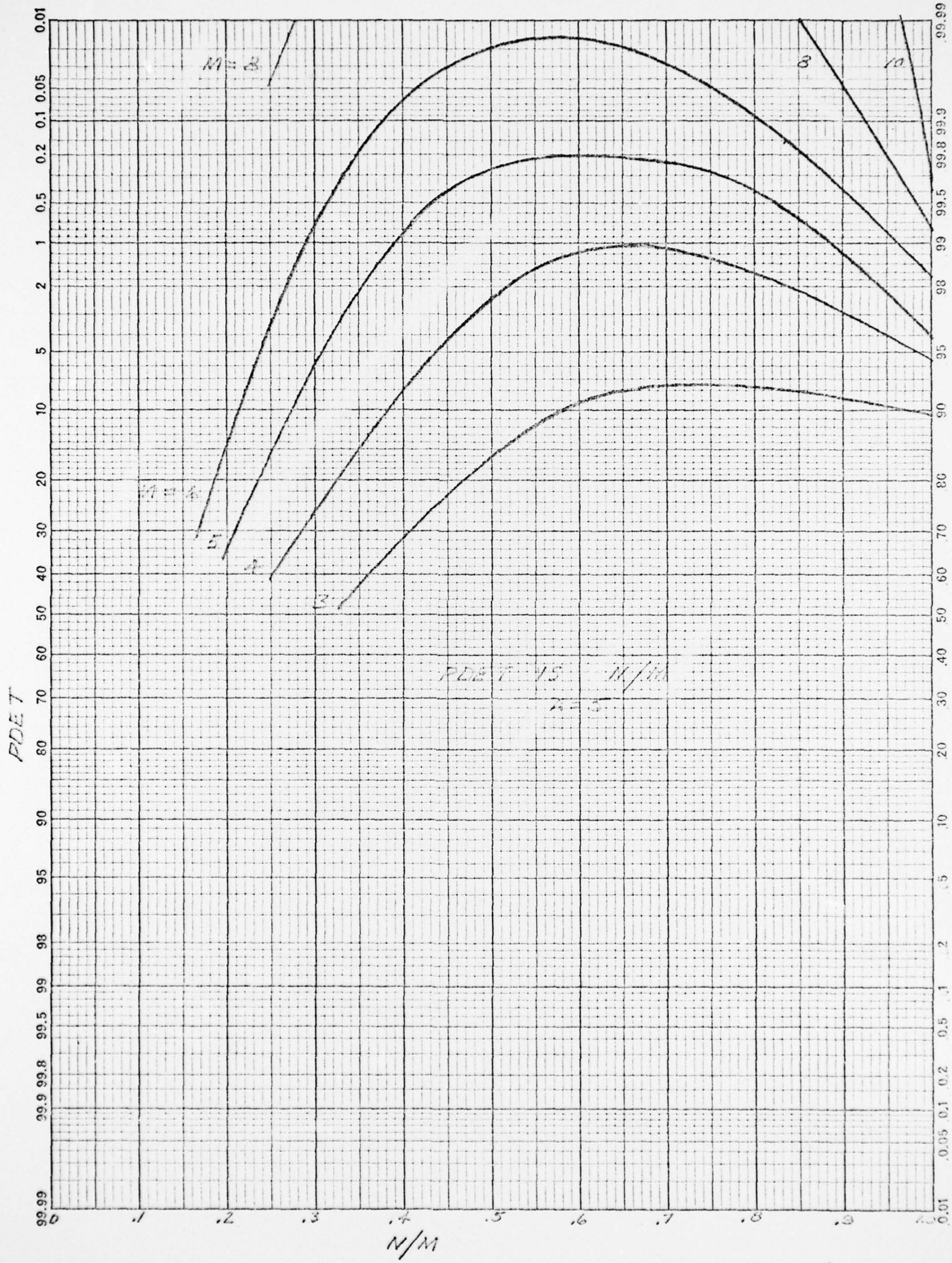
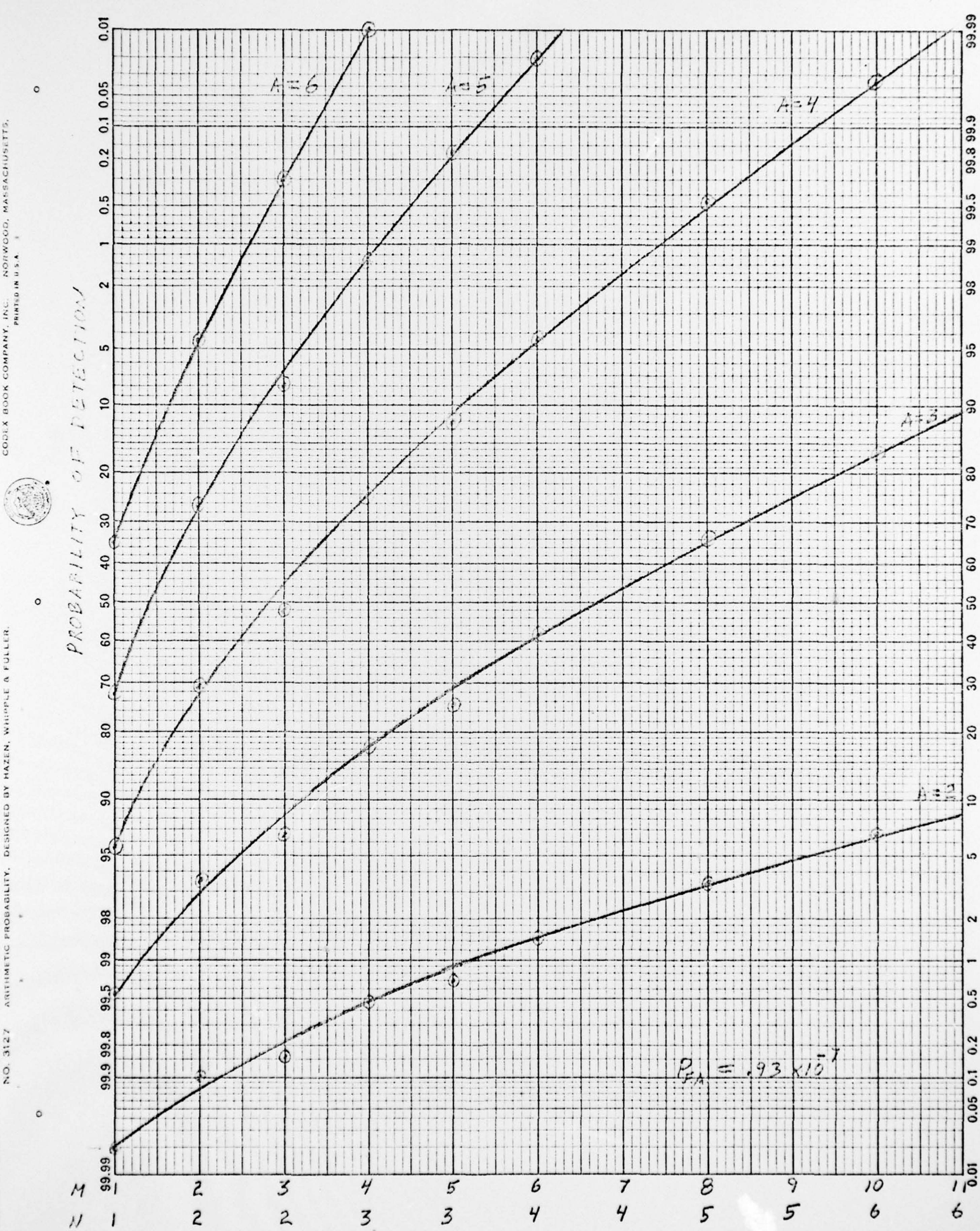


Figure 4

NO. 3127 ARITHMETIC PROBABILITY, DESIGNED BY HAZEN, WHIPPLE & FULLER.
CODEX BOOK COMPANY, INC. NORWOOD, MASSACHUSETTS, PRINTED IN U.S.A.
M=4
M=3
PDET
N/M
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
0.01
0.05
0.1
0.2
0.5
1
2
5
10
20
30
40
50
60
70
80
90
95
98
99
99.5
99.8
99.9
99.99



NO. 3127 ARITHMETIC PROBABILITY. DESIGNED BY HAZEN, WHIPPLE & FULLER.
CODEX BOOK COMPANY, INC. NORWOOD, MASSACHUSETTS. PRINTED IN U.S.A.
PROBABILITY OF DETECTION
A=6
A=5
A=4
A=3
A=2
$P_{FA} = .93 \times 10^{-7}$
99.99 99.9 99.8 99.5 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0.01 0.05 0.1 0.2 0.5 1 2 5 10 20 30 40 50 60 70 80 90 95 98 99 99.5 99.8 99.9 99.99
M 1 2 3 4 5 6 7 8 9 10 11
N 1 2 2 3 3 4 4 5 5 6 6

Figure 6

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11. "ETRAN Programmers Guide," Code 3110D, U.S. Navy Electronics Laboratory, Jan 1967.

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 LIB          ETRANF
 TRA          1000
C                                            0
C                                      SYS-PRØC*CUMULATIVE PRØBABILITY
C                                      PRØCEDURE*CØMPUTE
C                                      CALL*INVERSE
C                                      PFA1=PFA
C                                      VX=PFA1
C                                      CALL*CUMULAT
C                                      CUMPFA1=CUM
C                                      PFA=PFA1*(0.0000001/CUMPFA1)**(1/NN)
C                                      VX=PFA
C                                      CALL*CUMULAT
C                                      CUMPFA=CUM
C                                      AA=1
C                          ASET        NØ-ØP
C                                      CALL*PRØBDET
C                                      PRØBSP=PRØB
C                                      VX=PRØB
C                                      CALL*CUMULAT
C                                      PRØBDCUM=CUM
C                                      CALL*PUTITØUT*1000
C                                      AA=AA+1
C                                      WHEN*AA//LTEQ//15//THEN*GØTØ*ASET
C                                      RETURN
C                                      END-PRØC*CØMPUTE
C                                      PRØCEDURE*INVERSE
C                                      HH=(2*NN-1)*(2*MM-2*NN+1)/MM
C                                      RR=4*(2*NN-MM-1)/(3*(2*NN-1)*(2*MM-2*NN+1))
C                                      WW=(5.2*(HH+4)**.5-(RR*(7.25*HH-1)))/HH
C                                      PFA=NN/(NN+(MM-NN+1)*EE**(2*WW))
C                                      RETURN
C                                      END-PRØC*INVERSE
C                                      PRØCEDURE*CUMULAT
C                          ALPHA       II=NN
C                                      CUM=0
C                          BETA        CUM=PC(II)*VX**II*(1-VX)**(MM-II)+CUM
C                                      II=II+1
C                                      WHEN*II//LTEQ//MM//THEN*GØTØ*BETA
C                                      RETURN
C                                      END-PRØC*CUMULAT
C                                      PRØCEDURE*PRØBDET
C                                      TSTAR=(-RAT2LGE*LØG(PFA))**0.5
C                                      SUM=0
C                                      PN=1
C                                      WHEN*SNRFLAG//EQ//1//THEN*GØTØ*CALCSNR
C                                      AA=10**(SNR/20)
C                                      GØTØ*CALCFØZ
C                          CALCSNR     NØ-ØP
C                                      SNR=20*LØG(AA)
C                          CALCFØZ     NØ-ØP
C                                      WHEN*AA//GT//TSTAR+2.5//THEN*GØTØ*C1
C                                      WHEN*TSTAR//GT//AA+2.5//THEN*GØTØ*C2
C                                      DELZ=DELZØ
C                                      ZZ=TSTAR
C                          REGS        NØ-ØP
C                                      ZSQ=ZZ*ZZ
C                                      AZ=AA*ZZ
C                                      ASQ=AA*AA
```

```
C                      RATZA=ZZ/AA
C                      CN=-(ZSQ+ASQ-2*AZ)/2
C                      CM=1+1/(8*AZ)+9/(128*AZ*AZ)
C                      FØZ=INVRT2PI*CM*RATZA**0.5*EE**CN
C                      PART2=FØZ*DELZ
C                      SUM=SUM+PART2
C                      WHEN*|PART2/PART1|//GT//RN//THEN*GØTØ*SETZ
C                      DELZ=DELZ*2
C                      RN=RN/2
C          SETZ        ZZ=ZZ+DELZ
C                      PART1=PART2
C                      WHEN*|PART2/SUM|//GT//.0001//THEN*GØTØ*REGS
C                      PRØB=SUM
C                      RETURN
C          Q1          NØ-ØP
C                      DIFF=AA-TSTAR
C                      RAT=TSTAR/AA
C                      PRØB=1-INVRT2PI*RAT**0.5*EE**(-DIFF*DIFF/2)/DIF
C                      RETURN
C          Q2          NØ-ØP
C                      DIFF=TSTAR-AA
C                      RAT=TSTAR/AA
C                      PRØB=INVRT2PI*RAT**0.5*EE**(-DIFF*DIFF/2)/DIFF
C                      RETURN
C                      END-PRØC*PRØBDET
C                      END-DATA
```

```
PARAM'
MM'10'
NN'4'
BC'1'10'45'120'210'252'210'120'45'10'1'
SNRFLAG'1'
E'2.718281828459'
RAT2LGE'4.605172614960'
INVRT2PI'.398942280401'
PART1'1'
DELZO'
0.01'
OUTPUT'
NN'MM'AA'SNR'PFA'CUMPFA'PROBSP'PROBDCUM'
PAUSE
```